

# COMPLETE FAMILIES OF COMMUTING FUNCTIONS FOR COISOTROPIC HAMILTONIAN ACTIONS

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## INTRODUCTION

In this paper, we study symplectic and Poisson algebraic varieties. Let us start with the main definitions.

**Definition 1.** Let  $\mathcal{A}$  be a commutative associative algebra equipped with an additional anticommuting operation  $\{ , \} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  called a *Poisson bracket* such that

$$\begin{aligned} \{ab, c\} &= a\{b, c\} + \{a, c\}b, \\ \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} &= 0, \end{aligned}$$

for all  $a, b, c \in \mathcal{A}$ . Then  $\mathcal{A}$  is called a *Poisson algebra*. An ideal of a Poisson algebra is said to be *Poisson* if it is invariant with respect to the Poisson bracket; a homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  of Poisson algebras is said to be *Poisson* if  $\varphi(\{x, y\}) = \{\varphi(x), \varphi(y)\}$  for all  $x, y \in \mathcal{A}$ . The *Poisson centre* of  $\mathcal{A}$  is the set  $\{a \in \mathcal{A} \mid \{a, \mathcal{A}\} = 0\}$ .

Let  $M$  be an irreducible affine variety defined over a field  $\mathbb{F}$  of characteristic zero. Denote by  $M(\overline{\mathbb{F}})$  the set of its points over the algebraic closure of  $\mathbb{F}$ . As usual,  $\mathbb{F}[M]$  and  $\mathbb{F}(M) := \text{Quot } \mathbb{F}[M]$  stand for the algebras of regular and rational functions on  $M$ , respectively. They can be considered as subalgebras of  $\overline{\mathbb{F}}(M(\overline{\mathbb{F}}))$ , where  $\overline{\mathbb{F}}[M(\overline{\mathbb{F}})] = \mathbb{F}[M] \otimes \overline{\mathbb{F}}$ . All subvarieties of  $M$ , all differential forms on  $M$ , and all morphism of  $M$  are supposed to be defined over  $\mathbb{F}$ .

Suppose there is a non-degenerate closed regular 2-form  $\omega$  on the smooth locus of  $M$ . Then  $\omega$  induces a Poisson bracket on  $\mathbb{F}(M)$ . The variety  $M$  is said to be *symplectic* if  $\mathbb{F}[M]$  is a Poisson subalgebra of  $\mathbb{F}(M)$ , i.e., if  $\{\mathbb{F}[M], \mathbb{F}[M]\} \subset \mathbb{F}[M]$ . In particular, this is always the case for normal affine varieties. Set  $2n := \dim_{\mathbb{F}} M = \text{tr.deg } \mathbb{F}(M)$ . A family of functions  $\{f_1, \dots, f_n\} \subset \mathbb{F}(M)$  such that  $\{f_i, f_j\} = 0$  for all  $i$  and  $j$  is said to be *complete* if the  $f_i$ 's are algebraically independent.

The simplest example of a symplectic variety is an even-dimensional vector spaces  $V$  equipped with a non-degenerate skew-symmetric bilinear form  $\omega$ . Each Lagrangian decomposition  $V = V_+ \oplus V_-$  gives us a complete family of linear functions on  $V$ , namely, one has to take a basis for  $V_+$ . Another familiar example is the cotangent bundle of a smooth irreducible affine variety  $Y$ ,  $M = T^*Y$ , equipped with the canonical symplectic

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structure. Here  $\mathbb{F}[Y]$  is a Poisson commutative subalgebra of  $\mathbb{F}[M]$ , i.e.,  $\{\mathbb{F}[Y], \mathbb{F}[Y]\} = 0$ . Since  $\dim M = 2 \dim Y$ , the subalgebra  $\mathbb{F}[Y]$  contains a complete family of functions on  $M$ .

It is a challenging open problem to prove that for each affine symplectic variety  $M$  the Poisson algebra  $\mathbb{F}[M]$  contains a complete family.

Suppose  $h \in \mathbb{F}[M]$ . Let  $\eta_h$  be a vector field on the smooth locus of  $M$  uniquely defined by the formula  $dh = \omega(\eta_h, \cdot)$ . Then  $\eta_h$  defines a Hamiltonian dynamical systems on  $M$ , and any function  $f$  on  $M$  such that  $\{h, f\} = 0$  is called a *first integral* of this system. The intersection of the level hypersurfaces of first integrals is stable with respect to the flow generated by  $\eta_h$ . Thus, to understand dynamical properties of  $\eta_h$ , it is desirable to construct as many independent first integrals as possible. The triple  $(M, \omega, h)$  is said to be *completely integrable* if there are algebraically independent first integrals  $f_1, \dots, f_n$  such that  $\{f_i, f_j\} = 0$  and  $2n = \dim M$ .

**Definition 2.** An irreducible affine algebraic variety  $X$  is said to be *Poisson* if  $\mathbb{F}[X]$  is a Poisson algebra.

Any Poisson structure on  $\mathbb{F}[X]$  uniquely extends to  $\mathbb{F}(X)$ .

Let  $G$  be a connected linear algebraic group over  $\mathbb{F}$  with  $\mathfrak{g} := \text{Lie } G$ . Recall that  $\mathcal{S}(\mathfrak{g}) = \mathbb{F}[\mathfrak{g}^*]$  equipped with the Kirillov bracket is a Poisson algebra, i.e.,  $\mathfrak{g}^*$  is a Poisson variety.

**Definition 3.** An algebraic action  $G \times X \rightarrow X$  is said to be *Hamiltonian* if there is a  $G$ -equivariant map, called *the moment map*,  $\pi : X \rightarrow \mathfrak{g}^*$  such that  $\pi^* : \mathcal{S}(\mathfrak{g}) \rightarrow \mathbb{F}[X]$  is a Poisson homomorphism.

Each function in  $\pi^*(\mathcal{S}(\mathfrak{g}))$  is called a *Noether integral* on  $X$ . Their most important property is given by the Noether theorem:  $\{\mathbb{F}(X)^G, \pi^*(\mathcal{S}(\mathfrak{g}))\} = 0$ . Our goal is to construct a complete family of Noether integrals on a Hamiltonian symplectic  $G$ -variety  $M$ . To this end, we have to study the closure  $\overline{\pi(M)} \subset \mathfrak{g}^*$ , which is a Poisson variety, not necessarily symplectic.

Let  $X \subset \mathfrak{g}^*$  be an irreducible  $G$ -invariant subvariety. Then  $X$  is a zero locus of a  $G$ -invariant prime ideal  $I \triangleleft \mathcal{S}(\mathfrak{g})$ . Being  $G$ -invariant means that  $\{\mathfrak{g}, I\} \subset I$ . In other words  $I$  is a Poisson ideal. Hence  $\mathbb{F}[X]$  is a Poisson algebra (a Poisson quotient of  $\mathcal{S}(\mathfrak{g})$ ) and  $X$  is a Poisson variety. Set

$$l(X) := (\dim X + \text{tr.deg } \mathbb{F}(X)^G)/2.$$

It follows from Rosenlicht's theorem, that  $l(X) = \dim X - \frac{1}{2} \max_{\gamma \in X(\overline{\mathbb{F}})} \dim(\mathfrak{g}\gamma)$ , where  $X(\overline{\mathbb{F}}) \subset (\mathfrak{g} \otimes \overline{\mathbb{F}})^*$ . Let  $R \subset \mathbb{F}[X]$  be a Poisson commutative subalgebra. Take  $\gamma \in X(\overline{\mathbb{F}})$  such that the  $G$ -orbit  $G\gamma$  is of maximal possible dimension. A subspace  $\langle d_\gamma a \mid a \in R \rangle \subset T_\gamma^* X \subset \mathfrak{g}$ , generated by the differentials, is isotropic with respect to the symplectic form  $\hat{\gamma}(x, y) = \gamma([x, y])$  (here  $x, y \in \mathfrak{g}$ ). Hence the dimension of this subspace is less or equal to  $l(X)$  and also  $\text{tr.deg } R \leq l(X)$ . A family  $\{f_1, \dots, f_{l(X)}\} \subset \mathbb{F}[X]$  is said to be *complete* if  $\{f_i, f_j\} = 0$  for all  $i, j$  and  $f_1, \dots, f_{l(X)}$  are algebraically independent.

The following is the main result of this paper.

**Theorem 1.** *Let  $I$  be a  $G$ -invariant prime ideal of  $S(\mathfrak{g})$  and  $X = \text{Spec}(S(\mathfrak{g})/I)$  the corresponding Poisson variety. Then there are functions  $f_1, \dots, f_{l(X)} \in S(\mathfrak{g})$  such that their images  $\overline{f_i}$  in  $S(\mathfrak{g})/I$  form a complete family on  $X$ .*

In case  $X = \mathfrak{g}^*$  the existence of a complete family in  $\mathbb{F}[\mathfrak{g}^*] = S(\mathfrak{g})$  was conjectured by Mishchenko and Fomenko [13], and proved by Sadetov [15]. A clearer treatment of this result is given by Bolsinov [2]. Our proof of Theorem 1 follows the same strategy as the proofs of Sadetov and Bolsinov for  $S(\mathfrak{g})$ . First we establish this result in case of reductive  $G$  (see Section 2). In the general case, we argue by induction on  $\dim G$  (see Section 4).

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## 1. APPLICATIONS OF THEOREM 1

Let  $(M, \omega)$  be a symplectic affine variety and  $G \times M \rightarrow M$  a Hamiltonian algebraic action. Write  $M(\overline{\mathbb{F}})^{sm}$  for the smooth locus of  $M(\overline{\mathbb{F}})$ . For each  $x \in M(\overline{\mathbb{F}})^{sm}$  denote by  $(\mathfrak{g}x)^\perp$  the orthogonal complement of  $\mathfrak{g}x$  taken with respect to  $\omega$ . The function  $x \mapsto \dim(\mathfrak{g}x \cap (\mathfrak{g}x)^\perp)$  is constant on an open subset  $U \subset M(\overline{\mathbb{F}})^{sm}$  and its value  $d$  on  $U$  is said to be *the defect* of the action  $G \times M \rightarrow M$  (see [16, Chapter 2, §3]).

**Definition 4.** The *corank* of  $G \times M \rightarrow M$ , denoted by  $\text{cork} M$ , is defined by the formula

$$\text{cork} M := \min_{x \in M(\overline{\mathbb{F}})^{sm}} \dim(\mathfrak{g}x)^\perp - d.$$

In other words, it equals to the rank of the form  $\omega|_{(\mathfrak{g}x)^\perp}$  for generic  $x \in M(\overline{\mathbb{F}})^{sm}$ . A Hamiltonian action of  $G$  on a symplectic variety  $M$  is said to be *coisotropic* if  $\text{cork} M = 0$ , i.e., if  $(\mathfrak{g}x)^\perp \subset \mathfrak{g}x$  for generic  $x \in M(\overline{\mathbb{F}})^{sm}$ .

**Theorem 2.** *Let  $G \times M \rightarrow M$  be a coisotropic Hamiltonian action and  $\pi : M \rightarrow \mathfrak{g}^*$  the corresponding moment map. Then there are functions  $f_1, \dots, f_n \in S(\mathfrak{g})$ , where  $n = \dim M/2$ , such that  $\{\pi^*(f_1), \dots, \pi^*(f_n)\}$  is a complete family on  $M$ .*

*Proof.* Set  $X := \overline{\pi(M)}$ . Let  $x \in M(\overline{\mathbb{F}})$ . Because  $\pi^*$  is a  $G$ -equivariant Poisson homomorphism, the kernel of  $d\pi_x$  coincides with  $(\mathfrak{g}x)^\perp \otimes \overline{\mathbb{F}}$ . Therefore,  $\dim X = \max_{x \in M(\overline{\mathbb{F}})} \dim(\mathfrak{g}x)$  and  $\max_{\gamma \in X(\overline{\mathbb{F}})} \dim(\mathfrak{g}\gamma) = \max_{x \in M(\overline{\mathbb{F}})} \dim(\mathfrak{g}x) - d$ . Choose  $x \in M(\overline{\mathbb{F}})$  such that  $\dim(\mathfrak{g}x)$  is maximal. Then

$$l(X) = \dim(\mathfrak{g}x) - \frac{1}{2}(\dim(\mathfrak{g}x) - d) = (\dim(\mathfrak{g}x) + d)/2 = (\dim M - \text{cork} M)/2.$$

Clearly,  $2l(X) = \dim M$  if and only if the action  $G \times M \rightarrow M$  is coisotropic. By virtue of Theorem 1, there is a complete family  $\{f_i\}$  in  $\mathbb{F}[X]$ . Since  $\text{cork} M = 0$  and  $\pi^*$  is a Poisson homomorphism,  $\{\pi^*(f_i)\}$  is a complete family on  $M$ .  $\square$

**Corollary.** *A Hamiltonian action  $G \times M \rightarrow M$  is coisotropic if and only if there is a complete family of Noether integrals on  $M$ ; or, equivalently, each  $G$ -invariant Hamiltonian system on  $M$  is completely integrable in the class of Noether integrals.*

**Theorem 3.** *Let  $G \times M \rightarrow M$  be a Hamiltonian action with  $\text{cork } M = 2$ . Then there is a complete family in  $\mathbb{F}(M)$ . If in addition generic  $G$ -orbits on  $M$  are separated by regular invariants, then there is a complete family in  $\mathbb{F}[M]$ .*

*Proof.* Set  $X = \overline{\pi(M)}$ . Then  $l(X) = \dim M/2 - 1$ . By Theorem 1, there are functions  $f_1, \dots, f_{l(X)} \in \mathcal{S}(\mathfrak{g})$  such that their restrictions to  $X$  form a complete family. Set  $R := \pi^*(\mathcal{S}(\mathfrak{g}))$ . Let  $\langle d_x R \rangle$  be the subspace of  $T_x^* M(\overline{\mathbb{F}})$  generated by all differentials  $d_x f$  with  $f \in R$ . Since  $(\mathfrak{g}x)^\perp$  is the kernel of  $d\pi_x$ , we have  $\langle d_x R \rangle = \text{Ann}((\mathfrak{g}x)^\perp)$ . By Rosenlicht's theorem, generic  $G$ -orbits in  $M$  are separated by rational invariants. Therefore,  $\langle d_x(\mathbb{F}(M)^G) \rangle = \text{Ann}(\mathfrak{g}x)$  for generic  $x \in M(\overline{\mathbb{F}})$ . Since the action  $G \times M \rightarrow M$  is not coisotropic,  $(\mathfrak{g}x)^\perp \not\subset (\mathfrak{g}x)$  and there is at least one  $h \in \mathbb{F}(M)^G$  such that functions  $\{h, \pi^*(f_1), \dots, \pi^*(f_{l(X)})\}$  are algebraically independent. Recall that  $\{\mathbb{F}(M)^G, R\} = 0$ . Thus  $\{h, \pi^*(f_1), \dots, \pi^*(f_{l(X)})\}$  is a complete family on  $M$ . If generic  $G$ -orbits on  $M$  are separated by regular invariants, then  $\mathbb{F}(M)^G = \text{Quot } \mathbb{F}[M]^G$  and we can choose  $h$  in  $\mathbb{F}[M]^G$ .  $\square$

Let us say a few words about cotangent bundles. It was already mentioned that a complete family always exists here. But the construction of Theorems 1 and 2 provides other examples of complete families, which can be useful for other Hamiltonian systems.

Suppose that  $M = T^*X$ , where  $X$  is a  $G$ -variety. Then  $M$  possesses a canonical  $G$ -invariant symplectic structure such that the action of  $G$  is Hamiltonian. If the action  $G \times M \rightarrow M$  is coisotropic, then  $X$  has an open  $G$ -orbit [5]. For reductive  $G$  one can say more.

Suppose  $\mathbb{F}$  is algebraically closed and  $G$  is reductive. By a result of Knop [8], the action of  $G$  on  $T^*X$  is coisotropic if and only if a Borel subgroup  $B$  of  $G$  has an open orbit on  $X$ . Normal varieties having an open  $B$ -orbit are said to be *spherical*. It was known before that if  $X$  is spherical and  $X = G/H$ , where  $H$  is a reductive subgroup of  $G$ , then each  $G$ -invariant Hamiltonian system on  $T^*X$  is integrable within the class of Noether integrals, see [11], [7]. Here we extend this result to all affine spherical  $G$ -varieties. Smooth affine spherical varieties are classified (under mild technical constraints) in [9]. It would be interesting to study complete families on their cotangent bundles.

Other well-studied coisotropic actions on cotangent bundles are related to the so called *Gelfand pairs*. Suppose  $\mathbb{F} = \mathbb{R}$  and  $M = T^*X$ , where  $X = G/K$  is a Riemannian homogeneous space. Then  $X$  is called *commutative* or the pair  $(G, K)$  is called a *Gelfand pair* if the action  $G \times M \rightarrow M$  is coisotropic. Gelfand pairs can be characterised by the following equivalent conditions.

- (1) The algebra  $\mathcal{D}(X)^G$  of  $G$ -invariant differential operators on  $X$  is commutative.
- (2) The algebra of  $K$ -invariant measures on  $X$  with compact support is commutative with respect to convolution.
- (3) The representation of  $G$  on  $L^2(X)$  has a simple spectrum.

Theorem 2 and its corollary provide two more equivalent conditions.

- (4) There is a complete family of Noether integrals on  $T^*X$ .

- (5) Each  $G$ -invariant Hamiltonian system on  $M$  is completely integrable in the class of Noether integrals.

According to [16], if  $G/K$  is a Gelfand pair and  $G = N \ltimes L$  is a Levi decomposition of  $G$  such that  $K \subset L$ , then  $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$  and  $\mathfrak{n}$  is at most two-step nilpotent. These conditions guarantee that the construction of a complete family on  $\pi(M)$  would have at most three induction steps. Thus, one can hope for precise formulas for our commuting families and applications to physical problems. Gelfand pairs are partially classified in [18] and completely in [19].

## 2. THE REDUCTIVE CASE

In this section,  $G$  is a connected reductive algebraic group. Here one can apply a very powerful tool, the so called “argument shift method”. It was used by Manakov [10], Mishchenko and Fomenko [12], and Bolsinov [1] in constructions of complete families on  $\mathfrak{g}^*$  and coadjoint  $G$ -orbits. The reader is referred to [6, Chapter 4] for a thorough exposition and historical remarks. Let us briefly outline this method.

Let  $r$  be the rank of  $\mathfrak{g}$ . Choose any set  $F_1, \dots, F_r$  of free generators of  $\mathbb{F}[\mathfrak{g}^*]^G$ . For any  $a \in \mathfrak{g}^*$ , let  $\mathcal{F}_a$  denote a finite set  $\{\partial_a^k F_i \mid k \geq 0, i = 1, \dots, r\} \subset \mathcal{S}(\mathfrak{g})$ . Then  $\{\mathcal{F}_a, \mathcal{F}_a\} = 0$ , see e.g. [14, Sections 1.12, 1.13]. Here we should mention that this fact is stated for  $\mathbb{F} = \mathbb{C}$  in [14], but the proofs are valid over all fields of characteristic zero.

Recall that the *index of a Lie algebra*  $\mathfrak{g}$  is the minimum of dimensions of stabilisers  $\mathfrak{g}_\xi$  over all covectors  $\xi \in \mathfrak{g}^*$ , i.e.,  $\text{ind } \mathfrak{g} = \min_{\xi \in \mathfrak{g}^*} \dim \mathfrak{g}_\xi$ .

**Theorem 4.** [1, Theorem 2] *Suppose  $\mathfrak{g}$  is a complex reductive Lie algebra and  $\xi \in \mathfrak{g}^*$ . Then there is  $a \in \mathfrak{g}^*$  such that the restriction of  $\mathcal{F}_a$  to the coadjoint orbit  $G\xi$  contains  $\frac{1}{2} \dim(G\xi)$  algebraically independent functions if and only if  $\text{ind } \mathfrak{g}_\xi = \text{ind } \mathfrak{g}$ .*

**Conjecture (Elashvili).** If  $\mathfrak{g}$  is reductive and  $\xi \in \mathfrak{g}^*$ , then  $\text{ind } \mathfrak{g}_\xi = \text{ind } \mathfrak{g}$ .

For the classical Lie algebras, Elashvili’s conjecture is proved in [17] under the assumption that  $\text{char } \mathbb{F}$  is good for  $\mathfrak{g}$ . W. de Graaf used a computer program to verify the conjecture for the exceptional Lie algebras, see [4]. Unfortunately, a case-free proof proposed by J.-Y. Charbonnel [3] contains a gap in the final part, Lemma 5.11. A conceptual proof of Elashvili’s conjecture is highly desirable.

Let  $\hat{V}_{a,\xi}$  be a linear subspace of  $T_\xi^*(\mathfrak{g}^*)$  generated by the differentials  $\{d_\xi F \mid F \in \mathcal{F}_a\}$  and  $V_{a,\xi}$  the restriction of  $\hat{V}_{a,\xi}$  to  $T_\xi(G\xi) = \mathfrak{g}_\xi$ . Since the orbit  $G\xi$  is a symplectic variety and the subspace  $V_{a,\xi}$  is isotropic, we get  $2 \dim V_{a,\xi} \leq \dim(G\xi)$ . The restriction of  $\mathcal{F}_a$  to  $G\xi$  contains a complete family if and only if there is  $a' \in Ga$  such that  $2 \dim V_{a',\xi} = \dim G\xi$ .

Assuming that Elashvili’s conjecture is true, we get

**Proposition 1.** *Let  $\mathfrak{g}$  be a complex reductive Lie algebra. Then for each  $\xi \in \mathfrak{g}^*$  there is  $a \in \mathfrak{g}^*$  such that  $2 \dim V_{a,\xi} = \dim G\xi$ .*

**Lemma 1.** *The statement of Proposition 1 holds over any algebraically closed field  $\overline{\mathbb{F}}$  of characteristic zero.*

*Proof.* Let us fix a  $\mathbb{Q}$ -form  $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ . It exists because  $\mathfrak{g}$  is reductive and the ground field is assumed to be algebraically closed. Set  $\mathfrak{g}(\mathbb{C}) := \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{C}$ . Choose a set of generators  $\{F_1, \dots, F_r\} \subset \mathcal{S}(\mathfrak{g}_{\mathbb{Q}})^G$ . Since  $\mathbb{C}[\mathfrak{g}(\mathbb{C})^*]^G = \mathbb{Q}[\mathfrak{g}_{\mathbb{Q}}^*]^G \otimes \mathbb{C}$  the polynomials  $F_1, \dots, F_r$  are also generators of  $\mathcal{S}(\mathfrak{g}(\mathbb{C}))^G$ , and by the similar reason of  $\mathcal{S}(\mathfrak{g})^G$ . Take any  $\xi \in \mathfrak{g}(\mathbb{C})^*$ . According to Proposition 1, there is  $a \in \mathfrak{g}(\mathbb{C})^*$  such that  $2\dim V_{a,\xi} = \dim G\xi$ . The family  $\mathcal{F}_a$  depends on  $a \in \mathfrak{g}(\mathbb{C})^*$  regularly. Therefore condition  $2\dim V_{a,\xi} = \dim G\xi$  holds for the elements of an open subset  $U \subset \mathfrak{g}(\mathbb{C})^*$ . Since the intersection  $U \cap \mathfrak{g}_{\mathbb{Q}}^*$  is non-empty, there is  $a \in \mathfrak{g}_{\mathbb{Q}}^*$  with this property.

Assume that there is  $y \in \mathfrak{g}^*$  such that  $\dim Gy = 2m$  and  $\dim V_{a,y} < m$  for each  $a \in \mathfrak{g}_{\mathbb{Q}}^*$ . Set  $Y_a := \{\xi \in \mathfrak{g}_{\mathbb{Q}}^* \mid \dim V_{a,\xi} < m\}$  and  $P_m := \{\xi \in \mathfrak{g}_{\mathbb{Q}}^* \mid \dim(G\xi) \leq 2m - 2\}$ . These are closed subvariety of  $\mathfrak{g}_{\mathbb{Q}}^*$  and  $y \in Y_a(\overline{\mathbb{F}})$  for all  $a \in \mathfrak{g}_{\mathbb{Q}}^*$ . As we have just seen  $\bigcap_{a \in \mathfrak{g}_{\mathbb{Q}}^*} Y_a(\mathbb{C}) = P_m(\mathbb{C})$ . Hence the equality holds over any extension of  $\mathbb{Q}$ , in particular over  $\overline{\mathbb{F}}$ . Thus  $y \in P_m(\overline{\mathbb{F}})$ . A contradiction.  $\square$

*Proof of Theorem 1 in the reductive case.* Let  $I$  be a prime Poisson ideal of  $\mathcal{S}(\mathfrak{g})$  and  $X(\overline{\mathbb{F}})$  a closed subvariety of  $(\mathfrak{g} \otimes \overline{\mathbb{F}})^*$  defined by  $I$ .

Choose a set of homogeneous generators  $\{F_1, \dots, F_r\} \subset \mathbb{F}[\mathfrak{g}^*]^G$ . Let  $\hat{F}_i$  denote the restriction of  $F_i$  to  $X$ . Each fibre of the quotient morphism  $\mathfrak{g}^* \rightarrow X//G$  contains finitely many  $G$ -orbits. Hence for generic  $\xi \in X(\overline{\mathbb{F}})$  the differentials  $\{d_{\xi}\hat{F}_i \mid i = 1, \dots, r\}$  generate a subspace of dimension  $d := \dim X - \dim(G\xi)$ . According to Lemma 1, there is an element  $a \in (\mathfrak{g} \otimes \overline{\mathbb{F}})^*$  such that the restriction of  $\mathcal{F}_a$  to  $G(\overline{\mathbb{F}})\xi$  contains a complete family, i.e.,  $2\dim V_{a,\xi} = \dim(G(\overline{\mathbb{F}})\xi)$ . There is an open subset of such elements. In particular, we may (and will) assume that  $a \in \mathfrak{g}^*$ . Then  $\mathcal{F}_a$  is a subset of  $\mathcal{S}(\mathfrak{g})$ . Each differential  $d_{\xi}\hat{F}_i$  is zero on  $\mathfrak{g}\xi$ . Therefore

$$\dim \langle d_{\xi}f \mid f \in \mathcal{F}_a|_X \rangle = d + \dim(G\xi)/2 = (d + \dim X)/2 = l(X)$$

and the restriction of  $\mathcal{F}_a$  to  $X$  contains a complete family.  $\square$

### 3. AUXILIARY RESULTS

In this section we collect several facts concerning structural properties of algebraic Lie algebras. They will be used in the proof of the main theorem.

**Definition 5.** A  $(2n+1)$ -dimensional Lie algebra  $\mathfrak{h}$  with  $n \geq 0$  is a *Heisenberg algebra* if there is a basis  $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$  for  $\mathfrak{h}$  such that  $[x_i, x_j] = [y_i, y_j] = 0$ ,  $[\mathfrak{h}, z] = 0$ , and  $[x_i, y_j] = \delta_{ij}z$ .

**Lemma 2.** Suppose  $\mathfrak{n}$  is a nilpotent Lie algebra such that each commutative characteristic ideal of  $\mathfrak{n}$  is one-dimensional. Then  $\mathfrak{n}$  is a Heisenberg algebra.

*Proof.* Let  $\mathfrak{z}$  be the centre of  $\mathfrak{n}$ . Then  $\dim \mathfrak{z} = 1$ . Consider the upper central series of  $\mathfrak{n}$

$$\mathfrak{z} = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \dots \subset \mathfrak{n}_{k-1} \subset \mathfrak{n}_k = \mathfrak{n},$$

i.e.,  $\mathfrak{n}_i/\mathfrak{n}_{i-1}$  is the centre of  $\mathfrak{n}/\mathfrak{n}_{i-1}$ . The centre of  $\mathfrak{n}_1$  is a commutative characteristic ideal of  $\mathfrak{n}$ . Hence, it is one-dimensional (coincides with  $\mathfrak{z}$ ) and  $\mathfrak{n}_1$  is a Heisenberg algebra. Let

$\mathfrak{z}_n(\mathfrak{n}_1)$  be the centraliser of  $\mathfrak{n}_1$  in  $\mathfrak{n}$ . Clearly,  $\mathfrak{z}_n(\mathfrak{n}_1)$  is an ideal in  $\mathfrak{n}$ . We claim that  $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{z}_n(\mathfrak{n}_1)$  and  $\mathfrak{n}_1 \cap \mathfrak{z}_n(\mathfrak{n}_1) = \mathfrak{z}$ . Indeed, let  $\xi \in \mathfrak{n}$ . Then  $[\xi, \mathfrak{n}_1] \subset \mathfrak{n}_0$  and there is an element  $\xi_0 \in \mathfrak{n}_1$  such that  $[\xi - \xi_0, \mathfrak{n}_1] = 0$ .

Let  $\mathfrak{z}_0$  be the centre of  $\mathfrak{z}_n(\mathfrak{n}_1)/\mathfrak{z}$ . Since  $\mathfrak{n}/\mathfrak{z} = (\mathfrak{n}_1/\mathfrak{z}) \oplus (\mathfrak{z}_n(\mathfrak{n}_1)/\mathfrak{z})$  is the direct sum of two ideals,  $\mathfrak{z}_0$  lies in the centre of  $\mathfrak{n}/\mathfrak{z}$ . Thus,  $\mathfrak{z}_0 \subset (\mathfrak{n}_1/\mathfrak{z})$  and  $\mathfrak{z}_0 = 0$ . Since  $\mathfrak{z}_n(\mathfrak{n}_1)/\mathfrak{z}$  is a nilpotent Lie algebra, we have  $\mathfrak{z}_n(\mathfrak{n}_1)/\mathfrak{z} = 0$ , and  $\mathfrak{n} = \mathfrak{n}_1$  is a Heisenberg algebra.  $\square$

Let  $N$  be the unipotent radical of  $G$ . Set  $\mathfrak{n} := \text{Lie } N$ . For any action  $P \times Y \rightarrow Y$  let  $Y/P$  stand for the set of  $P$ -orbits on  $Y$ .

**Lemma 3.** *Suppose  $N$  is a Heisenberg group and the centre  $\mathfrak{z}$  of  $\mathfrak{n}$  lies in the centre of  $\mathfrak{g}$ . Given a non-zero  $\alpha \in \mathfrak{z}^*$ , set  $Y_\alpha := \{\gamma \in \mathfrak{g}^* \mid \gamma|_{\mathfrak{z}} = \alpha\}$ . Then  $Y_\alpha/N = \text{Spec } \mathbb{F}[Y_\alpha]^N$ ; the natural action of  $G/N$  on  $Y_\alpha/N$  is Hamiltonian and the moment map  $\phi : Y_\alpha/N \rightarrow (\mathfrak{g}/\mathfrak{n})^*$  is an isomorphism.*

*Proof.* Choose a Levi decomposition  $G = N \ltimes L$  and let  $V$  be an  $L$ -invariant complement of  $\mathfrak{z}$  in  $\mathfrak{n}$ . Set  $S_\alpha := \{\gamma \in \mathfrak{g}^* \mid \gamma(V) = 0, \gamma|_{\mathfrak{z}} = \alpha\}$ . In other words,  $S_\alpha = (\mathfrak{g}/\mathfrak{n})^* + \tilde{\alpha}$ , where  $\tilde{\alpha} \in \mathfrak{g}^*$ ,  $\tilde{\alpha}(V) = 0$ ,  $\tilde{\alpha}(\mathfrak{l}) = 0$ , and  $\tilde{\alpha}|_{\mathfrak{z}} = \alpha$ . Clearly  $S_\alpha \subset Y_\alpha$ . Each point  $\gamma \in Y_\alpha$  can be uniquely presented as a sum

$$\gamma = \beta + \text{ad}^*(\eta) \cdot \tilde{\alpha} + \tilde{\alpha}, \text{ where } \beta(\mathfrak{n}) = 0 \text{ and } \eta \in \mathfrak{n}.$$

$$\text{Thus } N\gamma \cap S_\alpha = \gamma - \text{ad}^*(\eta) \cdot \gamma + \frac{1}{2}(\text{ad}^*(\eta))^2 \cdot \gamma = \{pt\}.$$

Set  $\phi(\gamma) := N\gamma \cap S_\alpha$ . Then we obtain an isomorphism  $\phi : Y_\alpha/N \rightarrow (\mathfrak{g}/\mathfrak{n})^*$ . Therefore  $Y_\alpha/N$  is an algebraic variety (an affine space) and  $\mathbb{F}[Y_\alpha/N] = \mathbb{F}[Y_\alpha]^N$ . For the rest of the proof we fix the isomorphism  $(\mathfrak{g}/\mathfrak{n})^* \cong \mathfrak{l}^*$  given by the Levi decomposition  $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$  and the induced isomorphisms  $S_\alpha \cong (\mathfrak{l}^* + \tilde{\alpha}) \cong \mathfrak{l}^*$ .

Let us check that  $\phi$  is  $G$ -equivariant. It is enough to verify that  $\phi(l \cdot N\gamma) = l \cdot \phi(N\gamma)$  for  $l \in L$ . Without loss of generality, we may assume that  $\gamma \in S_\alpha$ . Then  $l\gamma \in S_\alpha$ ,  $(l \cdot N\gamma) \cap S_\alpha = l\gamma$ , and  $\phi(l \cdot N\gamma) = l\gamma - \tilde{\alpha} = l(\gamma - \tilde{\alpha}) = l \cdot \phi(N\gamma)$ .

It remains to prove that  $\phi^*$  is a homomorphism of the Poisson algebras  $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$  and  $\mathbb{F}[Y_\alpha]^N$ . We have to show that  $\{\phi^*(f_1), \phi^*(f_2)\} = \phi^*(\{f_1, f_2\})$  for all  $f_1, f_2 \in \mathcal{S}(\mathfrak{g}/\mathfrak{n})$ . Since both  $\phi^*(f_1)$  and  $\phi^*(f_2)$  are  $N$ -invariant, it is enough to check the equality on  $S_\alpha$ .

Let  $\gamma \in S_\alpha$ . Recall that we have identified  $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$  with  $\mathcal{S}(\mathfrak{l}) \subset \mathcal{S}(\mathfrak{g})$ . This identification gives us that  $\{f_1, f_2\}(\tilde{\alpha}) = 0$ , and, hence,

$$\{f_1, f_2\}(\gamma) = \{f_1, f_2\}(\gamma - \tilde{\alpha}) = \{f_1, f_2\}(\phi(N\gamma)) = \phi^*(\{f_1, f_2\})(\gamma).$$

Below we show that  $\{\phi^*(f_1), \phi^*(f_2)\}(\gamma)$  is equal to  $\{f_1, f_2\}(\gamma)$ . It is well-known that  $G\gamma$  is a symplectic leaf of  $Y_\alpha$  and  $\mathfrak{g}^*$ . Also  $L\gamma$  is a symplectic leaf of  $S_\alpha$ . We have  $T_\gamma(G\gamma) = T_\gamma(L\gamma) \oplus T_\gamma(N\gamma)$ , where  $T_\gamma(L\gamma) = \mathfrak{l}\gamma$  and  $T_\gamma(N\gamma) = \mathfrak{n}\gamma$  are orthogonal and  $T_\gamma(L\gamma) \subset T_\gamma S_\alpha$ . Let  $F_i$  be the restriction of  $\phi^*(f_i)$  to  $G\gamma$ . Since  $G\gamma$  is a symplectic leaf of  $\mathfrak{g}^*$ , we have  $\{\phi^*(f_1), \phi^*(f_2)\}(\gamma) = \{F_1, F_2\}(\gamma)$ . Clearly the functions  $F_1$  and  $F_2$  are  $N$ -invariant, hence  $d_\gamma F_i(\mathfrak{n}\gamma) = 0$ . Thus  $\{F_1, F_2\}(\gamma) = \{F_1|_{L\gamma}, F_2|_{L\gamma}\}(\gamma) = \{f_1, f_2\}(\gamma)$  and we are done.  $\square$

**Corollary.** *In the setting of Lemma 3, we have*

$$(\mathbb{F}[\mathfrak{g}^*][1/z])^N \cong \mathcal{S}(\mathfrak{g}/\mathfrak{n}) \otimes \mathbb{F}[z, 1/z] \subset \mathcal{S}(\mathfrak{g}/\mathfrak{n}) \otimes \mathbb{F}(\mathfrak{z}^*),$$

where  $z$  is a non-zero element of  $\mathfrak{z}$ . Moreover, if  $X \subset \mathfrak{g}^*$  is a closed  $G$ -invariant subvariety such that  $z|_X \neq 0$ , then  $(\mathbb{F}[X][1/z])^N$  is a Poisson quotient of  $S(\mathfrak{g}/\mathfrak{n}) \otimes \mathbb{F}[z, 1/z]$ .

*Proof.* Set  $X_\alpha := X \cap Y_\alpha$ . Then  $S_\alpha \cap X_\alpha$  defines a section of  $X_\alpha/N$ , i.e.,  $X_\alpha/N \cong X_\alpha \cap S_\alpha \cong \mathfrak{x}_\alpha$ , where  $\mathfrak{x}_\alpha \subset (\mathfrak{g}/\mathfrak{n})^*$  is a closed  $G$ -invariant (Poisson) subvariety. Therefore  $(\mathbb{F}[X][1/z])^N$  is a Poisson quotient of  $S(\mathfrak{g}/\mathfrak{n}) \otimes \mathbb{F}[z, 1/z]$ .  $\square$

*Remark 1.* From Lemma 3 one can deduce that  $\mathbb{F}(\mathfrak{g}^*)^G = \mathbb{F}((\mathfrak{g}/\mathfrak{n})^*)^{G/N} \otimes \mathbb{F}(\mathfrak{z}^*)$ . In particular, in this case  $\mathbb{F}(\mathfrak{g}^*)^G$  is a rational field.

Let  $H \triangleleft N$  be a connected commutative normal subgroup of  $G$  with  $\text{Lie } H = \mathfrak{h}$ .

**Lemma 4.** Fix  $\alpha \in \mathfrak{h}^*$  and let  $Y_\alpha$  be the preimage of  $\alpha$  under the natural restriction  $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ . Then  $Y_\alpha/H = \text{Spec } \mathbb{F}[Y_\alpha]^H$  and the restriction map  $\pi_\alpha : Y_\alpha \rightarrow (\mathfrak{g}_\alpha)^*$  defines an isomorphism  $Y_\alpha/H \cong (\mathfrak{g}_\alpha/\mathfrak{h})^* \times \{\alpha\}$ .

*Proof.* Let  $\xi \in \mathfrak{g}_\alpha$ ,  $h \in \mathfrak{h}$ ,  $\eta \in \mathfrak{g}$ . Then

$$(h \cdot \gamma)(\eta) = \gamma([\eta, h]) = \alpha([\eta, h]) = -(\eta \cdot \alpha)(h).$$

Therefore  $H\gamma = \gamma + \mathfrak{h}\gamma = \gamma + (\mathfrak{g}/\mathfrak{g}_\alpha)^*$  and each non-zero fibre of the natural  $G_\alpha$ -equivariant restriction  $\pi_\alpha : Y_\alpha \rightarrow (\mathfrak{g}_\alpha)^*$  is exactly one  $H$ -orbit. Let us fix a decomposition  $\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{m}$ . Choose any  $\tilde{\alpha} \in \mathfrak{g}^*$  such that  $\tilde{\alpha}(\mathfrak{m}) = 0$  and  $\tilde{\alpha}|_{\mathfrak{h}} = \alpha$ . Then  $Y_\alpha = (\mathfrak{g}/\mathfrak{h})^* + \tilde{\alpha}$  and  $\pi_\alpha(Y_\alpha) \cong (\mathfrak{g}_\alpha/\mathfrak{h})^* \times \{\tilde{\alpha}\} \cong (\mathfrak{g}_\alpha/\mathfrak{h})^* \times \{\alpha\}$ .  $\square$

Until the end of this section, we assume that  $\mathbb{F} = \overline{\mathbb{F}}$ . Suppose that  $X \subset \mathfrak{g}^*$  is a closed  $G$ -invariant subvariety. Let  $\mathfrak{x}_\mathfrak{h} \subset \mathfrak{h}^*$  denote the image of  $X$  under the restriction  $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ . Set  $\mathbb{K} := \mathbb{F}(\mathfrak{x}_\mathfrak{h})$  and

$$(1) \quad \hat{\mathfrak{g}} := \{\xi \in \mathfrak{g} \otimes \mathbb{K} \mid [\xi, \mathfrak{h}](\mathfrak{x}_\mathfrak{h}) = 0\},$$

$$(2) \quad \hat{\mathfrak{h}} := \{\xi \in \mathfrak{h} \otimes \mathbb{K} \mid \alpha(\xi(\alpha)) = 0 \text{ for each } \alpha \in \mathfrak{x}_\mathfrak{h} \text{ such that } \xi(\alpha) \text{ has sense}\}.$$

Then  $\hat{\mathfrak{g}}$  is the Lie algebra of all rational morphisms  $\xi : \mathfrak{x}_\mathfrak{h} \rightarrow \mathfrak{g}$  such that  $\xi(\alpha) \in \mathfrak{g}_\alpha$  whenever  $\xi(\alpha)$  is defined.

Since  $\mathfrak{h}$  is a commutative ideal of  $\mathfrak{g}$ , we have  $\mathfrak{h} \otimes \mathbb{K} \triangleleft \hat{\mathfrak{g}}$ . Moreover,  $\hat{\mathfrak{h}}$  is also an ideal of  $\hat{\mathfrak{g}}$ . The main object of our interest is the quotient Lie algebra  $\tilde{\mathfrak{g}} := \hat{\mathfrak{g}}/\hat{\mathfrak{h}}$ . Another way to define this Lie algebra is to say that  $\tilde{\mathfrak{g}} := \{\xi \in \mathfrak{g} \otimes_{\mathfrak{h}} \mathbb{K} \mid [\xi, \mathfrak{h}](\mathfrak{x}_\mathfrak{h}) = 0\}$ .

Set  $\mathcal{A} := (\mathbb{F}[X] \otimes_{\mathbb{F}[\mathfrak{x}_\mathfrak{h}]} \mathbb{K})^H = \mathbb{F}[X]^H \otimes_{\mathbb{F}[\mathfrak{x}_\mathfrak{h}]} \mathbb{K}$ . Then the algebra  $\mathcal{A}$  carries a natural Poisson structure induced from  $\mathbb{F}[X]$ .

**Lemma 5.** Suppose that  $\mathbb{F} = \overline{\mathbb{F}}$ . Then  $\mathcal{A}$  is a Poisson quotient of  $S(\tilde{\mathfrak{g}})$ .

*Proof.* Both algebras  $\mathcal{A}$  and  $S(\tilde{\mathfrak{g}})$  consist of rational functions on  $\mathfrak{x}_\mathfrak{h}$  with coefficients in  $\mathbb{K}[X]^H$  and  $S(\mathfrak{g})$ , respectively. Thus, it suffices to verify the claim at generic  $\alpha \in \mathfrak{x}_\mathfrak{h}$ .

Fix a vector space decomposition  $\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{m}$  and let  $s : \{\alpha\} \times (\mathfrak{g}_\alpha/\mathfrak{h})^* \rightarrow \text{Ann}(\mathfrak{m}) \subset Y_\alpha$  be the corresponding section of  $\pi_\alpha$ . Then  $S_\alpha := \text{Im } s$  is a closed subset of  $Y_\alpha$  and by Lemma 4,



$S_\alpha \cap X \cong \pi_\alpha(Y_\alpha \cap X) \cong (X \cap Y_\alpha)/H$ . Let  $\mathcal{A}_\alpha$  be the subset of functions that are defined at  $\alpha$ . Then for generic  $\alpha \in \mathfrak{x}_\mathfrak{h}$  we have a surjective map

$$\varepsilon_\alpha : \mathcal{A}_\alpha \rightarrow \mathbb{F}[Y_\alpha \cap X]^H \cong \mathbb{F}[S_\alpha \cap X].$$

On the other hand,  $\hat{\mathfrak{g}}(\alpha) := \{\xi(\alpha) \mid \xi \in \hat{\mathfrak{g}}\} = \mathfrak{g}_\alpha$  for generic  $\alpha \in \mathfrak{x}_\mathfrak{h}$ . The algebra  $\tilde{\mathfrak{g}}(\alpha)$ , defined in the same way, is a 1-dimensional central extension,  $(\mathfrak{g}_\alpha/\mathfrak{h}) \oplus \mathbb{F}w$ , of  $\mathfrak{g}_\alpha/\mathfrak{h}$  given by

$$[\xi + \mathfrak{h}, \eta + \mathfrak{h}] := ([\xi, \eta] + \mathfrak{h}) + \tilde{\alpha}[\xi, \eta]w, \quad \text{where } \xi, \eta \in \mathfrak{g}_\alpha,$$

and  $\tilde{\alpha} \in \mathfrak{g}_\alpha^*$  is a linear function such that  $\tilde{\alpha}|_\mathfrak{h} = \alpha$ . Hence  $(\tilde{\mathfrak{g}}(\alpha))^* = (\mathfrak{g}_\alpha/\mathfrak{h})^* \times \mathbb{F}\tilde{\alpha}$  and  $S_\alpha \cap X$  is a closed subset of  $(\tilde{\mathfrak{g}}(\alpha))^*$ .

Therefore  $\mathcal{A}$  is a quotient of  $s(\tilde{\mathfrak{g}})$ . Since the Poisson structure on  $\mathcal{A}$  is induced from  $\mathbb{F}[X]$  and  $X$  is a Poisson subvariety of  $\mathfrak{g}^*$ , it is indeed a Poisson quotient.  $\square$

*Remark 2.* Informally speaking,  $\mathcal{A}$  is the algebra of functions on the set  $\tilde{X}$  of all rational morphisms  $\psi : \mathfrak{x}_\mathfrak{h} \rightarrow X$  such that  $\psi(\alpha) \in (X \cap S_\alpha)$ . Here  $\tilde{X}$  is also a set of the  $H$ -invariant rational morphisms  $\psi' : \mathfrak{x}_\mathfrak{h} \rightarrow X$  such that  $\psi'(\alpha) \in (X \cap Y_\alpha)$ .

#### 4. PROOF OF THEOREM 1

Let  $I \triangleleft s(\mathfrak{g})$  be a  $G$ -invariant (i.e., Poisson) prime ideal and  $X \subset \mathfrak{g}^*$  a subvariety defined by  $I$ . Then  $\mathcal{P} := \mathbb{F}[X] = s(\mathfrak{g})/I$  is a Poisson algebra. In this section we construct a complete family on  $X$ .

Set  $n := \dim X = \text{tr.deg } \mathcal{P}$ ,  $d := \text{tr.deg } \mathbb{F}(X)^G$ . Then  $n - d$  is the dimension of a generic  $G$ -orbit on  $X$ , and  $l = l(X) = (n + d)/2$ . The task is to construct  $l$  functions  $f_i \in s(\mathfrak{g})$  such that  $\{f_i, f_j\} \in I$  and their restrictions to  $X$  are algebraically independent. We argue by induction on  $\dim \mathfrak{g}$ .

In case of reductive  $G$ , Theorem 1 is proved in Section 2. Assume therefore that  $G$  is not reductive. If  $I$  contains a non-trivial ideal  $\mathfrak{c} \triangleleft \mathfrak{g}$ , then  $X \subset (\mathfrak{g}/\mathfrak{c})^*$  and we can replace  $\mathfrak{g}$  by  $\mathfrak{g}/\mathfrak{c}$  without loss of generality. Below we assume that  $I \cap \mathfrak{g} = 0$ . Let  $\mathfrak{n}$  be the nilpotent radical of  $\mathfrak{g}$  and  $N \subset G$  the connected subgroup with  $\text{Lie } N = \mathfrak{n}$ .

- Suppose that  $\mathfrak{n}$  is a Heisenberg Lie algebra and  $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$  is a central subalgebra of  $\mathfrak{g}$ . Then Lemma 3 applies. Let  $z \in \mathfrak{z}$  be a non-zero element. Since  $\mathfrak{z} \not\subset I$ , we have  $z|_X \neq 0$ . Set  $\tilde{\mathcal{P}} := (\mathcal{P}[1/z])^N$ . By Lemma 3,  $\tilde{\mathcal{P}}$  is a Poisson quotient of  $s(\mathfrak{g}/\mathfrak{n}) \otimes \mathbb{F}[z, 1/z]$ . The Lie algebra  $\mathfrak{g}/\mathfrak{n}$  is reductive, therefore there are pairwise commuting functions  $f_1, \dots, f_m \in s(\mathfrak{g}/\mathfrak{n}) \otimes \mathbb{F}(\mathfrak{z}^*)$  such that their images form a complete family in  $\tilde{\mathcal{P}}$ . After multiplying by a common denominator, we may assume that each  $f_i \in s(\mathfrak{g}/\mathfrak{n})$ .

Choose a decomposition  $\mathfrak{n} = V_+ \oplus V_- \oplus \mathfrak{z}$ , where  $V_+$  and  $V_-$  are commutative subalgebras. Recall that  $\mathfrak{n}_\gamma = \mathfrak{z}$  for generic  $\gamma \in X$ . Therefore a generic  $(G/N)$ -orbit on  $\tilde{X} := \text{Spec } \tilde{\mathcal{P}}$  has dimension  $(n - d) - (\dim \mathfrak{n} - 1)$ . Hence  $l(\tilde{X}) = l(X) - (\dim \mathfrak{n} - 1)/2$  in case  $s(\mathfrak{z}) \cap I \neq 0$ ; and  $l(\tilde{X}) = l(X) - (\dim \mathfrak{n} + 1)/2$  otherwise. If  $s(\mathfrak{z}) \cap I = 0$ , then  $f_1, \dots, f_m$  together with a basis for  $V_+ \oplus \mathfrak{z}$  give us a complete commutative family on  $X$ . On the other hand, if  $s(\mathfrak{z}) \cap I \neq 0$ , then we add a basis for  $V_+$  to  $\{f_i\}$  and again obtain a complete family on  $X$ .

• If the previous case does not hold, then either  $\mathfrak{n}$  is a Heisenberg Lie algebra such that  $[\mathfrak{n}, \mathfrak{n}]$  is not a central subalgebra of  $\mathfrak{g}$ , or  $\mathfrak{n}$  contains a commutative characteristic ideal  $\mathfrak{h}$  such that  $\dim \mathfrak{h} > 1$ , see Lemma 2. In both cases, there is a commutative ideal  $\mathfrak{h} \subset \mathfrak{n}$  of  $\mathfrak{g}$  such that either  $[\mathfrak{g}, \mathfrak{h}] \neq 0$  or  $\dim \mathfrak{h} > 1$ . Set  $I_0 := I \cap \mathcal{S}(\mathfrak{h})$  and let  $\mathfrak{x}_{\mathfrak{h}}$  be the subvariety of  $\mathfrak{h}^*$  defined by  $I_0$ . By definition,  $\mathfrak{x}_{\mathfrak{h}}(\overline{\mathbb{F}})$  coincides with the image of the natural projection  $X(\overline{\mathbb{F}}) \rightarrow (\mathfrak{h} \otimes \overline{\mathbb{F}})^*$ .

Set  $\mathbb{K} := \mathbb{F}(\mathfrak{x}_{\mathfrak{h}})$ . Consider a Lie algebra  $\tilde{\mathfrak{g}} := \{\xi \in \mathfrak{g} \otimes_{\mathfrak{h}} \mathbb{K} \mid \{\xi, \mathfrak{h}\} \subset I_0 \otimes \mathbb{K}\}$ . Over algebraically closed field  $\tilde{\mathfrak{g}}$  coincides with the quotient  $\hat{\mathfrak{g}}/\hat{\mathfrak{h}}$ , where  $\hat{\mathfrak{g}}$  and  $\hat{\mathfrak{h}}$  are defined by Formulas (1) and (2). In the general case we have  $\tilde{\mathfrak{g}} \otimes \overline{\mathbb{F}} = \widehat{\mathfrak{g} \otimes \overline{\mathbb{F}} / \mathfrak{h} \otimes \overline{\mathbb{F}}}$ . Set  $\tilde{\mathcal{P}} := P^H \otimes_{\mathbb{F}[\mathfrak{x}_{\mathfrak{h}}]} \mathbb{K}$ . By Lemma 5,  $\tilde{\mathcal{P}} \otimes \overline{\mathbb{F}}$  is a Poisson quotient of  $\mathcal{S}(\tilde{\mathfrak{g}}) \otimes \overline{\mathbb{F}}$ . The Galois group  $\text{Gal}_{\mathbb{F}}(\overline{\mathbb{F}})$  of the field extension  $\mathbb{F} \subset \overline{\mathbb{F}}$  acts on both these Poisson algebras. By taking fixed points of  $\text{Gal}_{\mathbb{F}}(\overline{\mathbb{F}})$ , we conclude that  $\tilde{\mathcal{P}}$  is a Poisson quotient of  $\mathcal{S}(\tilde{\mathfrak{g}})$ .

We claim that  $\tilde{\mathcal{P}}$  contains no zero divisors. Indeed, suppose  $x, y \in \tilde{\mathcal{P}}$  and  $xy = 0$ . After multiplying  $x$  and  $y$  by suitable elements of the field  $\mathbb{F}(\mathfrak{x}_{\mathfrak{h}})$ , we may assume that  $x, y \in P^{\mathfrak{h}}$ . Since the ideal  $I$  is prime, we get  $x = y = 0$ .

Set  $\tilde{X} := \text{Spec } \tilde{\mathcal{P}}$ . Then  $\tilde{X} \subset \tilde{\mathfrak{g}}^*$  is Poisson subvariety defined over  $\mathbb{K}$ . Let us compute  $l(\tilde{X})$ .

Let  $k$  be the dimension of a generic  $H$ -orbit on  $X$ . Note that  $k$  is also the dimension of a generic  $G$ -orbit in  $\mathfrak{x}_{\mathfrak{h}}$ . Since  $\mathfrak{h}$  is an algebraic Lie algebra consisting of nilpotent elements, we have  $\mathbb{F}(X)^H = \text{Quot } \mathbb{F}[X]^H$ . Therefore generic  $H$ -orbits on  $X$  are separated by regular  $H$ -invariants and  $\text{tr.deg } P^{\mathfrak{h}} = n - k$ . Hence  $\text{tr.deg } \tilde{\mathcal{P}} = n - k - \dim \mathfrak{x}_{\mathfrak{h}}$ .

Next,  $\mathbb{K}(\tilde{X}) = \mathbb{F}(X)^H \otimes_{\mathbb{K}} \mathbb{K}$ . Recall that  $\tilde{X}$  is a Poisson subvariety of  $\tilde{\mathfrak{g}}^*$ . In particular, the Poisson centre  $\mathcal{Z}\mathbb{K}(\tilde{X})$  of  $\mathbb{K}(\tilde{X})$  coincides with  $\mathbb{K}(\tilde{X})^{\mathfrak{g}}$ . Because  $\mathfrak{h}$  is commutative,  $\mathfrak{h} \subset \mathbb{F}[X]^H$ . Therefore the Poisson centre  $\mathcal{Z}\mathbb{F}(X)^H$  is equal to the Poisson centraliser  $R := \{f \in \mathbb{F}(X) \mid \{f, \mathbb{F}[X]^H\} = 0\}$ . Clearly  $R$  contains both  $\mathbb{F}[\mathfrak{x}_{\mathfrak{h}}]$  and  $\mathcal{Z}\mathbb{F}(X) = \mathbb{F}(X)^{\mathfrak{g}}$ . For generic  $\gamma \in X(\overline{\mathbb{F}})$ , we have  $\dim(\mathfrak{h}|_{\mathfrak{g}\gamma}) = \dim(\mathfrak{h}\gamma) = k$ . Since all functions in  $\mathbb{F}(X)^{\mathfrak{g}}$  are constant on  $G$ -orbits, the subspace of  $T_{\gamma}^*X$  generated by  $d_{\gamma}\mathbb{F}[\mathfrak{x}_{\mathfrak{h}}]$  and  $d_{\gamma}(\mathbb{F}(X)^{\mathfrak{g}})$  has dimension  $d + k$ . Hence,  $\text{tr.deg } R \geq d + k$ . By a simple dimension reason  $\text{tr.deg } R = d + k$ . Since  $\mathcal{Z}\mathbb{K}(\tilde{X}) = \mathcal{Z}\mathbb{F}(X)^H \otimes_{\mathbb{K}} \mathbb{K}$ , we get  $\text{tr.deg } \mathbb{K}(\tilde{X})^{\mathfrak{g}} = d + k - \dim \mathfrak{x}_{\mathfrak{h}}$ . Thus

$$\begin{aligned} l(\tilde{X}) &= (\dim_{\mathbb{K}} \tilde{X} + \text{tr.deg. } \mathbb{K}(\tilde{X})^{\mathfrak{g}})/2 = (n - k - \dim \mathfrak{x}_{\mathfrak{h}} + d + k - \dim \mathfrak{x}_{\mathfrak{h}})/2 = \\ &= (n + d)/2 - \dim \mathfrak{x}_{\mathfrak{h}} = l(X) - \dim \mathfrak{x}_{\mathfrak{h}}. \end{aligned}$$

It remains to show that the dimension of  $\tilde{\mathfrak{g}}$  over  $\mathbb{F}(\mathfrak{x}_{\mathfrak{h}})$  is less than  $\dim_{\mathbb{F}} \mathfrak{g}$ . If this is not the case, then  $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{K}$  and  $\hat{\mathfrak{h}} = 0$  (here  $\hat{\mathfrak{h}}$  is the same as in (2)). From the first equality we get  $[\mathfrak{g}, \mathfrak{h}] \subset I_0$ , hence  $[\mathfrak{g}, \mathfrak{h}] = 0$ ; and by the second one  $\dim \mathfrak{h} = 1$ . Together these conclusions contradict the initial assumptions on  $\mathfrak{h}$ .

Applying the inductive hypothesis to  $\tilde{X}$ , we construct  $l(X) - \dim \mathfrak{x}_{\mathfrak{h}}$  functions  $\tilde{f}_i \in \mathcal{S}(\tilde{\mathfrak{g}})$  such that their restrictions give us a complete commutative family on  $\tilde{X}$ . After multiplying them by a suitable element of  $\mathbb{K}$ , we may assume that  $\tilde{f}_i \in \mathcal{S}(\mathfrak{g})$ . The remaining  $\dim \mathfrak{x}_{\mathfrak{h}}$  functions we get from  $\mathcal{S}(\mathfrak{h})$ .

Thus, Theorem 1 is proved.

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